Computational Simulation of Anisotropic Self-Morphing Materials in Architectural Design

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This study introduces a computational simulator designed for materials that morph due to internal stresses, applied to architectural contexts. This approach marks a significant evolution in architectural practices, highlighting a shift towards sustainability, adaptability, and responsiveness in design. These materials present new challenges in architectural design, necessitating advanced computational tools for form-finding to predict complex behaviors not easily inferred from initial conditions. Our simulator, integrated with Grasshopper and using the Kangaroo Physics plugin, aims to enhance shape-finding processes for these materials, providing reliable shape predictions and broadening design possibilities. Focusing on anisotropic materials, particularly fiber-based polymer composites, the simulator enables designers to create structures that can adapt to various conditions. This capability extends the potential for sustainable and innovative architectural solutions, moving beyond traditional design constraints to embrace the complexities of material behavior and interaction.
Utilizing sophisticated algorithms and models, the tool facilitates early simulation and visualization of materials and structures, bridging theoretical concepts with practical applications.

Keywords: *frustrated materials, material simulation, self-morphing, moldless fabrication, anisotropic materials, fiber composites*

INTRODUCTION

The architectural design field has seen significant evolution, moving from traditional methods to a modern approach emphasized by innovative material usage and precise applications. This transition towards computational design and material experimentation marks a notable development in the creation and construction of structures (Kolarevic, 2003). This paradigm shift reflects not only the changing aesthetic and functional demands of contemporary society but also underscores a dedication to sustainable and responsive design principles (Hensel and Menges, 2007). Architecture is set to become as adaptable and responsive as the environments for which it's designed.

The interpolation of computational design, applied physics, and architectural principles calls for early simulation and visualization of materials and structures, marrying theoretical concepts with practical application. Advanced tools, powered by sophisticated algorithms and physical models, have become valuable in the design process, allowing for the creation of structures that are more efficient, sustainable, and visually striking (Menges, 2012; Peters, 2013).

In this spectrum of design innovation, one notable approach involves leveraging self-morphing materials (Siefert *et al*., 2020; Blonder and Sharon, 2021). These materials are engineered to respond adaptively to internal stresses and can also be referred to as *frustrated materials* (Blonder and Sharon, 2020). These developments in self-morphing materials could transform the design of architectural facades. Future applications might involve creating composite surfaces that are initially flat but can morph into complex shapes on-site, eliminating the need for traditional molding. This method offers more design flexibility and reduces material waste by enabling adaptable and dynamic exteriors.

Material frustration occurs when materials experience internal conflicts due to differing responses to changes, whether internal or external. This is caused by their internal structure, leading them to a state where they cannot adopt a single stress-free configuration, but rather choose one that minimizes internal stresses, out of several possible configuration alternatives (Sharon and Efrati, 2010). This state can endow materials with unique properties, such as the ability to change shape or structural behavior under certain terms. Incorporating these materials into architectural designs presents challenges. Unlike traditional design processes where outcomes can be directly deduced from initial conditions, self-organizing systems introduce complex causality that is difficult to predict without advanced form-finding computational tools. Therefore, simulations become crucial in managing these complexities, offering a more reliable dependable approach than trial and error or educated guesses, making it an essential component of the design process.

Recent attempts have been made to address this issue, as evidenced by studies (Blonder and Sharon, 2021; Chen *et al*., 2022; Jourdan, 2023). Existing codes for forward or inverse design are not adapted, either because these are crafted for different type of material systems, or due to high accuracy and computational load, which not well-suited for designers. Others provide only a limited set of functionalities (Blonder and Sharon, 2021), indicating a gap between current capabilities and the needs of designers.

The simulator presented here builds upon the tool introduced by Blonder & Sharon (2021) and aims to bridge this gap. The tool, integrated into Grasshopper and leveraging the Kangaroo Physics plugin for physics simulations and optimization, is designed to improve shape-finding processes for self-morphing materials and offer reliable predictions of their final shapes.

We differentiate between isotropic materials, which have uniform properties in all directions, and anisotropic materials, whose properties vary by direction. Our focus in this study is on anisotropic materials, fiber-based polymer composite, which can be utilized in architectural and engineering applications for their engineered nature, customized to structural requirements, and exceptional ratio of strength to weight with high durability. This utilization of fiber-based composites in architecture not only enables the creation of innovative, fluid forms that challenge traditional aesthetics but also enhances building performance in terms of energy efficiency and environmental resilience. The versatility of composites allows for their integration in façades, structural components, and uniquely designed elements that contribute both functionally and visually to modern architecture.

This paper outlines the general structure and key design aspects of the tool in 'Computation Tool' section, followed by its validation against relevant physics theory, detailed in 'Evaluation' section. This concept extends to self-morphing materials in architecture, focusing on the complex interplay of anisotropic characteristics, which inherently encompasses isotropic properties as well. It enables the simulation and modeling of structures, allowing for greater control over their design and functionality. Utilizing this tool allows for the creation of architectural forms without relying on traditional molds by leveraging an enhanced understanding of material behavior in specific conditions, which in turn widens the range of design opportunities.

BACKGROUND

Theory of non-Euclidean Plates (NEP)

Recent advancements have led to the development of active materials programming techniques and theoretical models for understanding their distinctive properties. Key among these are incompatible sheets, like non-Euclidean plates (NEP) (Efrati *et al.* 2009; Sharon and Efrati, 2010) and shells (Armon *et al.*, 2011), characterized by their inherent non-Euclidean geometry. This inherent material structure can lead to the build-up of internal stresses and material frustration that result in the emergence of a surface of non-Euclidean geometry, highlighting a unique aspect of material science that explores the intrinsic spatial configurations of these materials. Within this theoretical framework, the intrinsic geometry of a solid is captured by the equilibrium distances among its constituents. This geometry is outlined by a reference metric field, $\bar{a}$, for in-plane distances and a reference curvature field, $\bar{b}$, indicating variations through the material's thickness. Such predefined geometries dictate unique 3D forms through mathematical and energetic analysis.

The material's state is determined by its actual metric a and actual curvature b. Deviations between the actual and reference geometries bring about energy costs, manifesting as either stretching or bending energy. The actual configuration of a material experiencing frustration finds a balance between its designed geometry, as prescribed by internal structure, and developing internal stresses, leading to an equilibrium of stretching and bending energies. Any modifications in shape or boundary cuts can lead to a new equilibrium state, causing a transformation in its 3D configuration. This final shape is influenced not only by the material's initial boundary shape but also by various factors including its lateral dimensions and thickness, highlighting the complexity of predicting the behavior of such materials. This theory elaborates on *material frustration* mentioned above, highlighting the connection between geometric variances and internal stresses. Such an understanding enables the creation of complex 3D designs from flat surfaces by adeptly adjusting distances, building on the foundational concept of material behavior under stress. Our simulation tool utilizes a spring model for energy minimization, making it well-suited for modeling systems that exhibit material frustration. This methodology aligns with the principle of minimizing energetic discrepancies to predict and shape the final form of frustrated materials.

The Material System

Composite materials is a family of materials comprised of advanced fibers such as carbon, glass, or aramid with polymers like polyester, epoxy, or vinyl-ester, forming fiber-reinforced polymers (FRP). This combination enhances mechanical properties, widely applied in various industries, especially construction (Mallick, 2007; Hollaway, 2010). Frustrated composites (FC), first introduced in (Blonder and Sharon, 2021), are based on fiber composites undergoing self-morphing process; these are formed by the construction of unbalanced laminates, purposely laminating pre-impregnated unidirectional fabric layers in different orientations (see example in Figure 1).

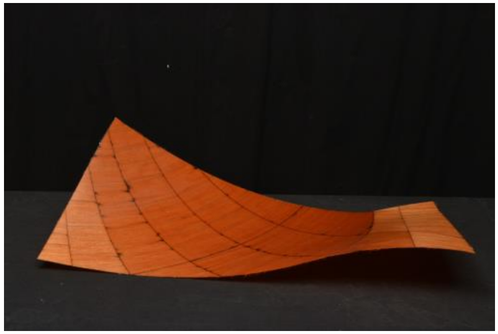


Figure 1
A sample of fiber-based composite from the exhibition 'Creative Differences' by 'Automorph Network' presented at the London Design Biennale 2023. ('Creative Differences', 2023). Photo: Haim Zinger

The composites’ curing process induces shrinkage in the resin (by approximately 3-5% for epoxy resin), while fiber shrinkage is negligible, and the shrinkage is uniaxial in a perpendicular direction to fiber orientation. Along with curing-induced shrinkage, variations in coefficient of thermal expansion upon cooling contribute to internal incompatibilities. This differential shrinkage, typically avoided in FRP by designing balanced laminates to eliminate internal torque (Nairn, 1985), is harnessed in FC to induce internal stresses, thus creating complex shapes. Instead of minimizing differential shrinkage, our strategy leverages it for creative design possibilities.

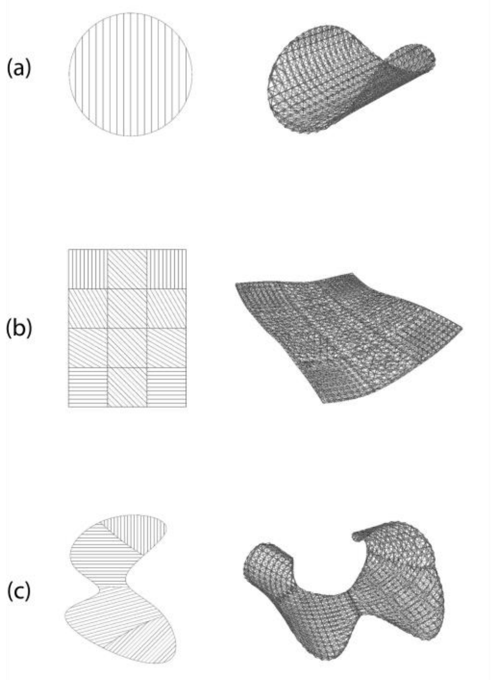


Figure 2
Illustrates surface segmentations and their varying fiber orientations.
For each surface segmentation, the first column details the boundary shape and fiber orientation of a given layer before simulation-driven morphing, while the second column displays the resultant shape post-simulation.
(a) Displays a plate-shaped surface with a consistent fiber orientation.
(b) Pixelated ribbon shape with diverse fiber orientations.
(c) Free-form shape with adaptive subdivision and a range of fiber orientations, referred to as 'Adaptive Pattern'.

This involves segmenting the surfaces to be created allowing each segment to differ in thickness and fiber orientations. This segmentation leads to various curvature amount and orientations, transitioning from uniform surfaces towards more complex shape formations. For examples of surface segmentations utilizing different fiber orientations, see Figure 2.

Existing Simulation Tools

While there are physical simulation tools available, they often do not meet the design community's requirements for creating complex shapes. The need for a design interface capable of generating complex surfaces based on theoretical principles is essential for moving into the architectural field. A recent simulation tool showed promise (Blonder and Sharon, 2021) but was limited in computational performance, accuracy, and design freedom. This referred tool elaborates on representing physical objects as a mesh of 'rods' in Rhinoceros 3D software through Grasshopper platform, simulating self-morphing materials using Kangaroo Physics plug-in for energy minimization. This tool models the lengths of 'rods' as springs with predefined properties, including axial and bending stiffness, to represent the rods' resistance to elongation and bending. It's important to note that these values are dimensionless and do not directly correspond to physical material data like Young's modulus, Poisson's ratio, or density. It then minimizes energy to find an optimal solution (Piker, 2019). However, the tool was limited to quad mesh with fiber orientations of 0, 45, and 90 degrees. Our aim is to extend these capabilities, offering a continuous range of fiber orientations, supporting various surface shapes and, including free-form options for an accessible and enabling interface for the designer. With that, we lay the groundwork for a simulation tool that supports varying thicknesses within a single input. The paper details the construct of the computational tool and describes its validation against theoretical principles.

COMPUTATIONAL TOOL

In this section, we detail our simulation tool, which integrates theoretical physics with computational methods, implemented through the Grasshopper platform and the Kangaroo Physics plugin.

From Physical Material to Geometrical Mesh

Our focus is on robust modeling both isotropic and anisotropic materials. However, our approach emphasizes capturing the behavior of these materials rather than replicating them. For isotropic materials, we use a pseudo-random mesh-based model to minimize orientation effects, identifying three key regions: top layer, middle web, and bottom layer. Connections between the top and bottom vertices form the middle web.

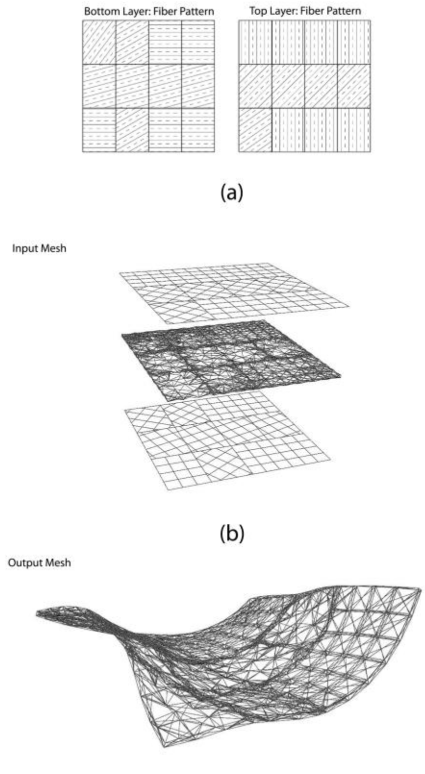


Figure 3
Illustrates the input and output mesh process.
(a) User-defined fiber patterns for top and bottom layers.
(b) Constructed mesh derived from the top and bottom layers' fiber patterns.
(c) Mesh after the self-morphing simulation.

In anisotropic modeling, we simulate the directional properties of fibers within composites on given surfaces, establishing mesh edges in orthogonal orientations to represent the fibers and matrix shrinkage directionality upon curing. These edges are then represented as rods in Kangaroo Physics for simulation. Connections between layers similarly create the middle structure, reflecting the composite's internal makeup.

This approach models materials to closely resemble their real-world behaviors. It utilizes a triangular mesh to mirror the randomness in isotropic materials and represents the directional fiber-matrix changes in anisotropic materials. This allows for the use of non-uniform meshes with various fiber orientations, enabling the simulation of complex structures as shown in Figure 3.

Generalization of Simulation Tool

Focusing on anisotropic mode we identify two distinct approaches within the anisotropic framework: the pixelization mode and the adaptive pattern mode. These two segmentation modes aim to mirror two different fabrication processes for complex surfaces, by sub-division of the panel into patches of different fiber orientation. In the pixelization mode, morphing is finely tuned by segmenting the surface into a uniform rectangular grid, each patch with its own fiber orientation, adjustable by the user. In the adaptive pattern mode, users have the opportunity to divide the input surface into non-regular regions, with each region assigned a different fiber orientation as part of a comprehensive optimization of the entire surface. The developed tool then constructs the mesh according to the panel's subdivided structure, facilitating forward simulation and providing design flexibility.

User Input

Across all modes, user can choose from three source shapes for a 2D surface: Plate, Ribbon, and Free-Form. Plate and Ribbon offer insights into material behavior under specific physics parameters, serving as baselines, while Free-Form is designed for more complex experiments. After selecting the source type, the user defines the shape, size, and thickness, then selects the mode (isotropic or anisotropic) and, if anisotropic, also chooses between pixelization or adaptive pattern.

For the isotropic mode, the user inputs shrinkage factors for both the top and bottom layers, along with axial and bending factors for all layers.

These factors can be calibrated against physical data and material properties, such as translating the shrinkage factor, axial stiffness, and bending stiffness of different materials into the rods' length factor, axial factor, and bending factor, applicable to both isotropic and anisotropic systems, as exemplified by (Dar *et al.*, 2024). In the pixelization mode, in addition to the physics factors mentioned for the isotropic mode, users can also set the pixelization resolution through the number of rows and columns in the grid forming the surface. Each pixel, or square (represented as a matrix entry), requires a specified fiber orientation by the user. In the adaptive pattern mode, beyond the physics factors and fiber orientation, user provide cutter curves instead of grid dimensions, which define how the source surface is segmented into irregular shapes, offering a customized approach to modeling.

Optimization Process

For the numerical simulation of self-morphed 2D surfaces, we utilize Rhinoceros 3D's parametric design tool, Grasshopper, incorporating the Kangaroo Physics plugin. This approach allows us to closely emulate the mechanical properties of a surface by adjusting its properties to minimize elastic energy. Kangaroo Physics typically models the linear elastic region of a material (Cheraud, 2020), but by integrating specific elasticity modulus values, including accurate relative bending and axial stiffness between rods representing fibre or matrix, we can significantly improve the precision of our simulations due to the detailed variation in material behaviors.

EVALUATION

To validate our simulation tool against theoretical principles, we selected four experimental setups as benchmarks, illustrated in Figure 4. The first setup, designed to validate the thickness-curvature relationship, is a two-layered ribbon with perpendicular fiber orientations—0 degrees for the bottom layer and 90 degrees for the top, measuring 4x10 cm with a thickness ranging from 0.1-1 cm. Here, the bottom layer is inert, showing no shrinkage, while the top layer experienced a 10% shrinkage rate. In the second experiment, the same

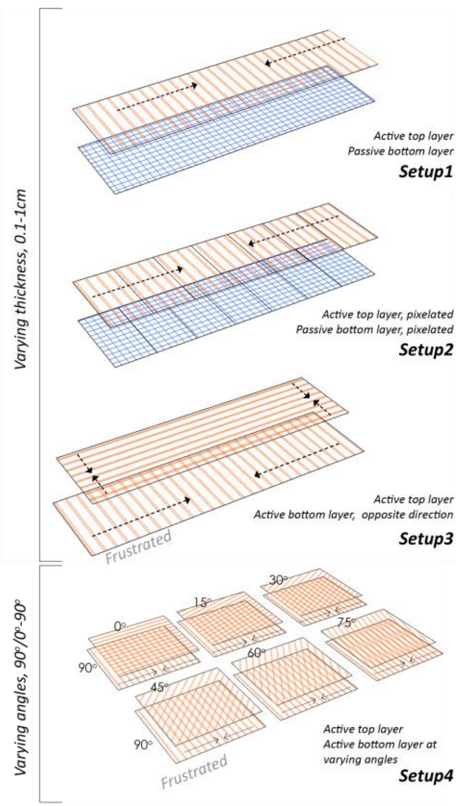
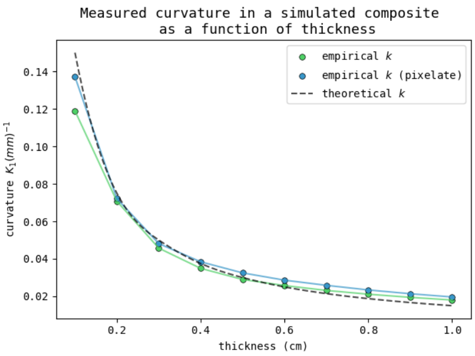
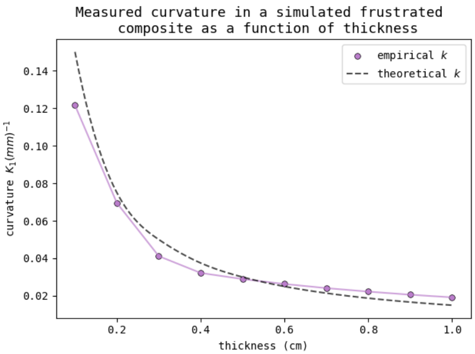


Figure 4
Top and bottom fiber patterns for four setups. Orange patterns are active layers with shrinkage; blue, passive layers without. Ribbon setups measure 4 x 10 cm; the square setup, 10 x 10 cm.

setup was used but the ribbon was divided into segments, maintaining identical fiber orientations. This verifies that the pixelization process does not affect the simulation outcomes. The third experiment mirrored the first, with the difference being that both layers now experienced the same shrinkage rate in perpendicular orientations. The final setup is a non-segmented, 10 x 10 cm square with a consistent 0.4 cm thickness and identical shrinkage rate across both layers. However, the top layer's fiber orientation varies from 0 to 90 degrees, contrasting with the bottom layer's fixed orientation of 0 degrees, for the validation of the relation between fiber angle and curvature.



(a)



(b)

Figure 5
The simulation tool evaluation results. (a) Measured principal curvature as a function of thickness in first (green) and second (blue) setups. Theoretical reference curvature is plotted in dashed line.
(b) Curvature of simulated frustrated ribbons as function of thickness.

Through these four setups, we analyze the radius of curvature and orientation of curvature to understand material behavior in this simulation tool. In our experiments, the actual curvature, k, in its principal direction, closely aligns with the reference curvature, $\bar{k}$, as a function of active deformation across the material's thickness, t. These deformations lead to a significant reference curvature and maintain flat reference metrics. The curvature's magnitude is influenced by the material's thickness and the differential strain, ε, representing the variation in deformation between the top and bottom layers. This is approximately expressed by the ratio: $\bar{k} \approx \varepsilon/t$ (Sharon and Efrati, 2010). Consequently, the final material shape is determined by energy minimization for a given reference curvature field, resulting in an approximate actual radius of curvature of $r = 1/\bar{k} \approx \varepsilon/t$.

Figure 5a illustrates the measured principal curvature as a function of thickness for $\varepsilon = 0.01$, aligning with typical material shrinking rates (3-5%) in our material systems. It also compares the theoretical expected principal curvature (Timoshenko, 1925). For this study, curvature is defined using a pre-factor of 1.5 for symmetrical bi-layer materials. However, for our initial two experiments, which aim to model edge cases with maximum curvature, we normalize our results with a factor of 0.23. This adjustment considers the consistency in expected trends, allowing for this factor to be determined through one-time calibration for specific material systems. The green and blue lines represent the measured curvature for the first and second setups, respectively, demonstrating the tool's reliability with both pixelized and non-pixelized inputs.

Figure 5b displays the curvature of simulated frustrated ribbons in relation to thickness from the third experiment. It differs from with theoretical reference curvature for anisotropic materials, showing that latent energy persists in frustrated composites.

Figure 6 visualizes the surfaces resulting from the second setup, showcasing curvature variations between a thickness of 0.1-1 cm.

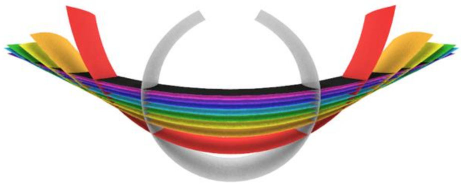


Figure 6
Illustration of curvature variations in the second setup between a thickness of 0.1 cm (maximum curvature, shown in white) and 1 cm (minimum curvature, shown in black).

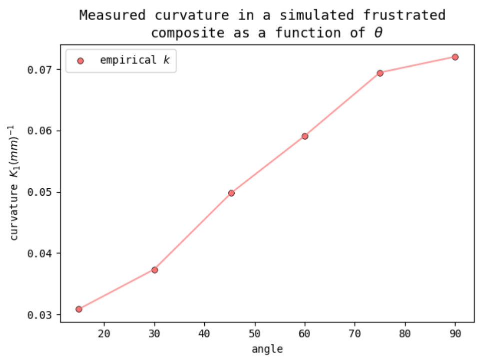


Figure 7
Measured curvature as a function of angle of simulated frustrated squares.

The curvature's magnitude in our study is determined by thickness and strain. Likewise, its orientation is influenced by the strain's principal direction and the material's anisotropic stiffness. In systems where anisotropy in stiffness is significant, the orientation of this stiffness influences curvature orientation, aligning perpendicularly to the fiber direction. Our theoretical analysis, supported by experimental evidence, suggests curvature is proportional to $\sin(\theta/2)$, where θ is the relative fiber angle between layers.

Figure 7 demonstrates the impact of anisotropy and strain direction on curvature orientation in simulated frustrated materials, as observed in the forth setup.

DISCUSSION AND CONCLUSION

This work introduces a new computational simulator for the exploration and design of self-morphing materials within architectural contexts. We've demonstrated how this tool, grounded in the principles of physics and computational design, and optimized through Grasshopper and Kangaroo Physics, offers a bridge between theoretical knowledge and practical architectural innovation.

Our findings reveal that the simulator's capacity to model anisotropic materials, especially fiber-based polymer composites, offers a new avenue for designing adaptive, efficient, and aesthetically appealing structures. Through the implementation of pixelization and adaptive pattern modes, the tool supports a broad range of design scenarios, from simple to complex surface shapes, facilitating a deeper integration of form, function, and materiality.

Moreover, the experiments conducted to validate the simulator's effectiveness underscore its reliability in simulating frustrated materials. By accurately capturing the intricate relationship between material properties and geometric configurations, the tool stands as a pivotal development in material innovation within architectural design.

Material parameters will be further calibrated in the simulation tool through physical experiments of complex surfaces, both pixelated and adaptive. Additional design features, such as variable thickness and material composition across the single panel will be developed for further design freedom. Additionally, ongoing work focuses on how the tool can generate quality data for solving inverse problems, specifically identifying initial conditions to achieve a desired 3D surface (Kapon, 2024).

In conclusion, the developed simulator represents a significant step forward in the integration of computational design with material science. It facilitates a deeper exploration of the potential of self-morphing materials, offering innovative solutions to contemporary architectural challenges. Our findings underscore the importance of simulation tools in the design process, enabling the realization of complex forms and behaviors that were previously unattainable.

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